Linear algebra cheat-sheet

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Matrix basics

A matrix is an array of numbers. $A \in \mathbb{R}^{m \times n}$ means that:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \qquad (m \text{ rows and } n \text{ columns})$$

Two matrices can be multiplied if inner dimensions agree:

$$\underset{(m \times p)}{C} = \underset{(m \times n)}{A} \underset{(n \times p)}{B} \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Transpose: The transpose operator A^T swaps rows and columns. If $A \in \mathbb{R}^{m \times n}$ then $A^\mathsf{T} \in \mathbb{R}^{n \times m}$ and $(A^\mathsf{T})_{ij} = A_{ji}$.

- $(A^\mathsf{T})^\mathsf{T} = A$.
- $(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}$.

Matrix basics (cont'd)

Vector products. If $x, y \in \mathbb{R}^n$ are column vectors,

- The **inner product** is $x^\mathsf{T} y \in \mathbb{R}$ (a.k.a. dot product)
- The **outer product** is $xy^\mathsf{T} \in \mathbb{R}^{n\times n}$.

These are just ordinary matrix multiplications!

Inverse. Let $A \in \mathbb{R}^{n\times n}$ (square). If there exists $B \in \mathbb{R}^{n\times n}$ with $AB = I$ or $BA = I$ (if one holds, then the other holds with the same B) then B is called the *inverse* of A, denoted $B = A^{-1}$.

Some properties of the matrix inverse:

- A^{-1} is unique if it exists.
- $(A^{-1})^{-1} = A$.
- $(A^{-1})^\mathsf{T} = (A^\mathsf{T})^{-1}$.
- $(AB)^{-1} = B^{-1}A^{-1}$.

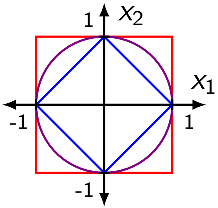
Vector norms

A norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying the properties:

- $\|x\| = 0$ if and only if $x = 0$ (definiteness)
- $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{R}$ (homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Common examples of norms:

- $\|x\|_1 = |x_1| + \cdots + |x_n|$ (the 1-norm)
- $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ (the 2-norm)
- $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ (max-norm)



norm ball: $\{x \mid \|x\| = 1\}$

Properties of the 2-norm (Euclidean norm)

- If you see $\|x\|$, think $\|x\|_2$ (it's the default)
- $x^\mathsf{T} x = \|x\|^2$
- $x^\mathsf{T} y \leq \|x\|\|y\|$ (Cauchy-Schwarz inequality)

Linear independence

A set of vectors $\{x_1, \ldots, x_n\} \in \mathbb{R}^m$ is **linearly independent** if

$$c_1 x_1 + \cdots + c_n x_n = 0 \quad \text{if and only if} \quad c_1 = \cdots = c_n = 0$$

If we define the matrix $A = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ then the columns of A are linearly independent if

$$Aw = 0 \quad \text{if and only if} \quad w = 0$$

If the vectors are not linearly independent, then they are **linearly dependent**. In this case, at least one of the vectors is redundant (can be expressed as a linear combination of the others). i.e. there exists a k and real numbers c_i such that

$$x_k = \sum_{i \neq k} c_i x_i$$

The rank of a matrix

$$\begin{aligned}\operatorname{rank}(A) &= \text{maximum number of linearly independent columns} \\ &= \text{maximum number of linearly independent rows}\end{aligned}$$

If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then

- $\operatorname{rank}(A) \leq \min(m, n)$
- $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B)) \leq \min(m, n, p)$
- if $\operatorname{rank}(A) = n$ then $\operatorname{rank}(AB) = \operatorname{rank}(B)$
- if $\operatorname{rank}(B) = n$ then $\operatorname{rank}(AB) = \operatorname{rank}(A)$

So multiplying by an invertible matrix does **not** alter the rank.

General properties of the matrix rank:

- $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$
- $\operatorname{rank}(A) = \operatorname{rank}(A^{\mathsf{T}}) = \operatorname{rank}(A^{\mathsf{T}}A) = \operatorname{rank}(AA^{\mathsf{T}})$
- $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\operatorname{rank}(A) = n$.

Linear equations

Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, linear equations take the form

$$Ax = b$$

Where we must solve for $x \in \mathbb{R}^n$. Three possibilities:

- No solutions. Example: $x_1 + x_2 = 1$ and $x_1 + x_2 = 0$.
- Exactly one solution. Example: $x_1 = 1$ and $x_2 = 0$.
- Infinitely many solutions. Example: $x_1 + x_2 = 0$.

Two common cases:

- **Overdetermined**: $m > n$. Typically no solutions. One approach is **least-squares**; find x to minimize $\|Ax - b\|_2$.
- **Underdetermined**: $m < n$. Typically infinitely many solutions. One approach is **regularization**; find the solution to $Ax = b$ such that $\|x\|_2$ is as small as possible.