TECHNIQUES FOR SOLVING DIFFERENTIAL EQUATIONS

Separable Equations	$\dfrac{dy}{dx} = g(x)f(y)$
Step 1: Separate x's and y's on different sides.	$\dfrac{1}{f(y)}dy = g(x)dx$
Step 2: Integrate both sides.	$\displaystyle\int \frac{1}{f(y)}dy = \int g(x)dx \ + \ C$
Step 3: Express y in terms of x where possible.	If $\lvert y\rvert = h(x)$, then $y = \pm h(x)$. If $y = \pm e^C\, h(x)$, then $y = A\, h(x)$ where A is *any* real number (including zero).
Step 4: Check that constant solutions $y = C$ where $f(C) = 0$ are not missed.	

First Order Linear Equations	$y' + P(x)y = Q(x)$
Step 1: Find integrating factor.	$I(x) = e^{\int P(x)dx}$
Step 2: Write differential equation as	$\big(I(x)y\big)' = I(x)Q(x)$
Step 3: Integrate both sides.	$\displaystyle I(x)y = \int I(x)Q(x)dx \ + \ C$
Step 4: Divide both sides by $I(x)$	$\displaystyle y = \frac{1}{I(x)}\Big(\int I(x)Q(x)dx \ + \ C\Big)$

Remark: Be careful with the sign of $P(x)$. For instance,
If $y' + \frac{1}{x}y = 1$, the integrating factor is $I(x) = e^{\int 1/x\, dx} = x$.
If $y' - \frac{1}{x}y = 1$, the integrating factor is $I(x) = e^{\int -1/x\, dx} = \frac{1}{x}$.

Second Order Linear Homogeneous Equations	$ay'' + by' + cy = 0$
Step 1: Write down the auxiliary equation.	$ar^2 + br + c = 0$
Step 2: Solve the auxiliary equation.	$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Step 3: Depending on the roots:	
(i) $b^2 - 4ac > 0$. Two real roots r_1, r_2.	$y = c_1\, e^{r_1 x} + c_2\, e^{r_2 x}$
(ii) $b^2 - 4ac = 0$. One real root $r = r_1 = r_2$.	$y = c_1\, e^{rx} + c_2\, x\, e^{rx}$
(iii) $b^2 - 4ac < 0$. Two complex roots $\alpha \pm i\beta$.	$y = c_1\, e^{\alpha x}\cos\beta x + c_2\, e^{\alpha x}\sin\beta x$

Second Order Linear Non-homogeneous Equations	$ay'' + by' + cy = G(x)$

Method of Undetermined Coefficients

Step 1:	Solve the complementary equation.	$ay_c'' + by_c' + cy_c = 0$
		$y_c = c_1\, y_1(x) + c_2\, y_2(x)$
Step 2:	Write down a trial solution:	
(i)	$G(x) = P(x)$	$y_p = Q(x)$
(i)	$G(x) = P(x)e^{kx}$	$y_p = Q(x)\, e^{kx}$
(i)	$G(x) = P(x)e^{kx}\cos mx$ or $P(x)e^{kx}\sin mx$	$y_p = Q(x)\, e^{kx}\cos mx + R(x)\, e^{kx}\sin mx$
	Here, $P(x), Q(x)$ and $R(x)$ are polynomials of the same degree.	
	Multiply y_p by x (or x^2) if one of the terms in the sum is $y_1(x)$ or $y_2(x)$.	
Step 3:	Substitute y_p into the differential equation, group terms of the same form together, e.g. $x^n e^{kx}\cos mx$, $x^n e^{kx}\sin mx$ and solve for the unknown coefficients.	$ay_p'' + by_p' + cy_p = G(x)$
Step 4:	Write down the general solution.	$y(x) = y_c(x) + y_p(x)$

Second Order Linear Non-homogeneous Equations	$ay'' + by' + cy = G(x)$

Method of Variation of Parameters

Step 1:	Solve the complementary equation.	$ay_c'' + by_c' + cy_c = 0$
		$y_c = c_1\, y_1 + c_2\, y_2$
Step 2:	The particular solution has the form:	$y_p = u_1\, y_1 + u_2\, y_2$
	Write down the two conditions:	$u_1'\, y_1 + u_2'\, y_2 = 0$ $u_1'\, y_1' + u_2'\, y_2' = G(x)/a$
	Solve the conditions for u_1' and u_2'.	$u_1' = \dfrac{G(x)y_2}{a(y_1'y_2 - y_2'y_1)} \quad u_2' = \dfrac{G(x)y_1}{a(y_2'y_1 - y_1'y_2)}$
Step 3:	Integrate u_1', u_2' to get u_1, u_2.	$u_1 = \int u_1' dx \; + \; c_1 \quad u_2 = \int u_2' dx \; + \; c_2$
Step 4:	Write down the general solution.	$y = (\int u_1' dx + c_1)y_1 \; + \; (\int u_2' dx + c_2)y_2$

APPLICATIONS OF DIFFERENTIAL EQUATIONS

ORTHOGONAL TRAJECTORIES	Given family of curves $y = f(k, x)$ 1. Write k in terms of y and x. 2. Differentiate, so k becomes 0. This is the diff eqn for the curves. 3. Replace y' by $-1/y'$. This is the diff eqn for the orth trajectories.	MIXING PROBLEMS $dy/dx =$ (rate in) $-$ (rate out) y is amount of salt at time t. (rate in) $=$ (vol in) $\times$ (conc in) (rate out) $= \dfrac{\text{vol out}}{\text{vol at time t}} \times y$

POPULATION MODELS — k relative growth rate, K carrying capacity, P_0 initial population

Description	Differential Equation	Solution	Eq. Solutions
Natural Growth	$\dfrac{dP}{dt} = kP,\ P(0) = P_0$	$P = P_0\, e^{kt}$	$P = 0$
Logistic Model	$\dfrac{dP}{dt} = kP(1 - \dfrac{P}{K}),\ P(0) = P_0$	$P = \dfrac{K}{1 + A\, e^{-kt}},\ A = \dfrac{K - P_0}{P_0}$	$P = 0, K$
Predator-Prey Systems	$\dfrac{dR}{dt} = kR - aRW$ $\dfrac{dW}{dt} = -rW + bRW$	Phase trajectories $\dfrac{dW}{dR} = \dfrac{-rW + bRW}{kR - aRW}$	$(R, W) = (0, 0)$ $(R, W) = (\frac{r}{b}, \frac{k}{a})$

SPRINGS & ELECTRIC CIRCUITS

$mx'' + cx' + kx = F(t)$	$LQ'' + RQ' + Q/C = E(t)$
x displacement	Q charge
dx/dt velocity	$dQ/dt = I$ current
m mass	L inductance
c damping constant	R resistance
k spring constant (force/extension)	$1/C$ elastance, C capacitance
$F(t)$ external force	$E(t)$ electromotive force

Description	Differential Equation	Solution
Simple Harmonic Motion	$mx'' + kx = 0$	$x(t) = c_1 \cos \omega t + c_1 \sin \omega t = A \cos(\omega t + \delta)$ frequency $\omega = \sqrt{\dfrac{k}{m}}$, period $T = \dfrac{2\pi}{\omega}$, amplitude $A = \sqrt{c_1^2 + c_2^2}$ phase angle δ, $\cos \delta = \dfrac{c_1}{A}$, $\sin \delta = -\dfrac{c_2}{A}$
Damped Vibrations	$mx'' + cx' + kx = 0$	$r = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ $c^2 - 4mk > 0$ overdamping $\quad x = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}$ $c^2 - 4mk = 0$ critical damping $\quad x = c_1\, e^{rt} + c_2\, te^{rt}$ $c^2 - 4mk < 0$ underdamping $\quad x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$
Forced Vibrations	$mx'' + cx' + kx = F(t)$	If $F(t)$ is periodic, then *resonance* occurs when the applied frequency ω_0 equals the natural frequency ω.

KEY ⋆ Definitions • Theorem ∘ Remark ⊗ Extra

9.1 Modeling with Differential Equations

⋆ differential equation, order.
solution, general solution.

⋆ equilibrium solution: a constant solution $y = C$.
set $y'' = y' = 0, y = C$ in diff eqn and solve for C.

⋆ initial condition, initial value problem.

9.2 Direction Fields and Euler's Method

⋆ direction field, solution curve.

⋆ autonomous differential equation $y' = f(y)$.
if $y = g(x)$ is a solution, so is $y = g(x + C)$.
e.g. natural growth, logistic model

∘ Graphical method:
1. draw direction field.
2. draw solution curve.

∘ Numerical method: Euler's method, step size h.
Solving $y' = F(x, y), y(x_0) = y_0$.
1. Set $x_n = x_0 + nh$ for $n \geq 1$.
2. Recursively, $y_{n+1} = y_n + hF(x_n, y_n)$ for $n \geq 0$.

9.3 Separable Equations

⋆ separable equations, orthogonal trajectories,
mixing problems (see formula sheet)

9.4 Population Models

∘ know how to derive solutions of natural growth/logistic model using separation of variables and partial fractions.

⋆ law of natural growth (see formula sheet)
compare with exponential decay $P' = -kP$
where k is negative, $P(t) = P_0 e^{-kt}$.

⋆ logistic differential equation (see formula sheet)
case 1: $0 < P_0 < K$. P increases and approaches K.
case 2: $P_0 > K$. P decreases and approaches K.
know how to see this from the differential equation.

⊗ natural growth with harvesting
$P' = kP - c, P(0) = P_0$
$P(t) - \frac{c}{k} = (P_0 - \frac{c}{k})e^{kt}$
trick: subs $y = P - \frac{c}{k}$ to get natural growth model.

⊗ logistic model with harvesting (see quiz 11)
$P' = kP(1 - \frac{K}{P}) - c, P(0) = P_0$
two equilibrium solutions $P_1, P_2 = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4c}{kK}}\right)$
case 1: $P_0 < P_1$. P approaches $-\infty$
case 2: $P_1 < P_0 < P_2$. P increases and approaches P_2.
case 3: $P_2 < P_0$. P decreases and approaches P_2.
trick: subs $y = P - P_1$ to get logistic model.

⊗ Seasonal growth $P' = kP\cos(rt - \phi)$, $P = Ce^{(k/r)\sin(rt-\phi)}$

⊗ Seasonal growth with harvesting $P' = kP\cos(rt - \phi) - c$

⊗ $P' = kP(1 - \frac{P}{K})(1 - \frac{m}{P})$, m extinction level.

9.5 First Order Linear Equations

⋆ first order linear equation (see formula sheet)

∘ to get unique solution from general solution:
initial value problem $y(0) = y_0$

9.6 Predator-Prey Systems

⋆ predator prey equations (see formula sheet)

⋆ phase plane, phase trajectory, phase portrait

⋆ know how to derive, draw and compare
phase trajectories and population graphs

⊗ $\frac{dW}{dR} = \frac{-rW + bRW}{kR - aRW} \Longrightarrow \frac{R^r W^k}{e^{bR} e^{aW}} = C$ (see S9.6Q7)

11.1 Second Order Linear Equations

⋆ second order linear equations
  $P(x)y'' + Q(x)y' + R(x) = 0$ homogeneous
  $P(x)y'' + Q(x)y' + R(x) = G(x)$ nonhomogeneous

⋆ linear combination, linearly independent solutions

• if differention equation is linear and homoegeneous, then linear combinations of solutions are solutions.

∘ $P(x)y'' + Q(x)y' + R(x) = G(x)$
to get unique solution from general solution:
1. initial condition, initial value problem
   $y(0) = y_0, y'(0) = y'_0$
   always has unique solution near $x = 0$
   if P, Q, R, G continuous and $P(0) \neq 0$
2. boundary condition, boundary value problem
   $y(0) = y_0, y(1) = y_1$
   may not have a solution

⋆ second order linear homogeneous equation, auxiliary/characteristic equation (see formula sheet)

11.2 Nonhomogeneous Linear Equations

⋆ complimentary equation $ay'' + by' + cy = 0$
complimentary solution $y_c(x)$
particular solution $y_p(x)$
general solution $y(x) = y_p(x) + y_c(x)$

⋆ method of undetermined coefficients (see formula sheet)

⋆ variation of parameters (see formula sheet)

11.3 Applications of Second Order Differential Equations

⋆ spring systems (see formula sheet)
  simple harmonic motion
  damped vibrations
    overdamping (returns to rest slowly)
    critical damping (returns to rest fastest)
    underdamping (oscillates before coming to rest)
    know how to draw graphs of above cases
    know how to check if graph cuts x-axis
  forced vibrations item[⋆] electric circuits (see formula sheet)

⋆ steady state solution
is behavior of solution as $t \to \infty$
see S17.3 example 3 note 1

⋆ a function $x(t)$ has period T if $x(t+T) = x(t)$ for all t.

11.4 Series Solutions

∘ finds solutions near $x = 0$ because we are writing the solution as a power series with center $x = 0$.

∘ step 1: write $y = \sum_{n=0}^{\infty} c_n x^n$.
step 2: differentiate to get y', y'' as power series.
step 3: substitute into differential equation. collect terms.
step 4: equate coefficients to get *recursion relations.*
step 5: solve recursion relations for small n to find patterns.
step 6: write down general solution.
  it will be in terms of certain coefficients
  e.g. c_0, c_1, which act as arbitrary constants.
step 7: (optional) recognize terms in general solution
  as power series of well-known functions.