With the aid of this and similar theorems the originals corresponding to various images can be established. In particular, the calculation of the original of every rational function can be reduced to the application of formula (2) (with a finite number of terms on its right-hand side).

Example 1. Let us find the original of the function $\omega/(p^2+\omega^2)$.

Using (2) and formulae for the calculation of residues, we get

$$f(t) = \operatorname*{res}_{p=\mathrm{i}\omega}\left[\frac{\omega\,\mathrm{e}^{pt}}{p^2+\omega^2}\right] + \operatorname*{res}_{p=-\mathrm{i}\omega}\left[\frac{\omega\,\mathrm{e}^{pt}}{p^2+\omega^2}\right] = \frac{\mathrm{e}^{\mathrm{i}\omega t}}{2\mathrm{i}} - \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}} = \sin\omega t,$$

in accordance with Tab. 28.2.

Extensive tables of transform pairs are given, e.g., in [118]. In such tables the image $F(p)$ is always given first (see Tab. 28.2).

To tables there is usually attached the so-called *grammar*, which summarizes the basic rules governing relationship between originals and images. In Tab. 28.3 below a sample of a grammar for Laplace transform is given.

TABLE 28.3

Image	Original
$\frac{1}{\alpha}F(\frac{p}{\alpha})$	$f(\alpha t)$
$pF(p) - f(0)$	$f'(t)$
$p^2F(p) - pf(0) - f'(0)$	$f''(t)$
$\frac{F(p)}{p}$	$\int_0^t f(\tau)\,\mathrm{d}\tau$
$\int_p^\infty F(r)\,\mathrm{d}r$	$\frac{f(t)}{t}$
$F(p-p_0)$	$\mathrm{e}^{p_0 t}f(t)$
$F(p)G(p)$	$\int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau$

Sufficient conditions for validity of the inversion formula for the Fourier transform are stated in the following assertion:

Theorem 5. *Let $f(t)$ be the original of a Fourier image $F(p)$. If $f(t)$ has bounded variation in a neighbourhood of a point t, then we have*

$$\frac{f(t+0)+f(t-0)}{2} = \lim_{\omega\to+\infty}\frac{1}{2\pi}\int_{-\omega}^{\omega} F(p)\,\mathrm{e}^{\mathrm{i}pt}\,\mathrm{d}p.$$

TABLE 28.2

$F(p) = \int_0^\infty f(t)\,e^{-pt}\,dt$	Original $f(t)$
$\frac{1}{p}$	1
$\frac{1}{p^2}$	t
$\frac{1}{p^{n+1}}$	$\frac{t^n}{n!}$, n a nonnegative integer
$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{(\pi t)}}$
$\frac{1}{p^{\nu+1}}$	$\frac{t^\nu}{\Gamma(\nu+1)}$, $\nu > -1$; for the Γ function see § 13.11
$\frac{1}{p+a}$	e^{-at}
$\frac{1}{(p+a)^2}$	$t\,e^{-at}$
$\frac{\omega}{p^2+\omega^2}$	$\sin \omega t$
$\frac{p}{p^2+\omega^2}$	$\cos \omega t$
$\frac{a}{p^2-a^2}$	$\sinh at$
$\frac{p}{p^2-a^2}$	$\cosh at$

If, in addition, $f(t)$ is continuous at the point t, then

$$\lim_{\omega \to +\infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} F(p)\,e^{ipt}\,dp = f(t).$$

If $f(t)$ is the original for the bilateral Laplace transform with image denoted by $F(p)$ and if

$$\int_{-\infty}^{\infty} \left| f(t)\,e^{-p_0 t} \right| dt < \infty$$

for some $p_0 = x_0 + iy_0$, then the function $f(t)\,e^{-x_0 t}$ is the original for the Fourier transform whose Fourier image is $F(x_0 + iy)$.

TABLE 28.2 (continued)

$F(p) = \int_0^{\infty} f(t)\,e^{-pt}\,dt$	Original $f(t)$
$\dfrac{e^{-ap}}{p}$	$\begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geqq a,\ a \geqq 0 \end{cases}$
$\dfrac{e^{-ap}}{\sqrt{p}}$	$\begin{cases} 0 & \text{for } t < a \\ \dfrac{1}{\sqrt{[\pi(t-a)]}} & \text{for } t \geqq a,\ a \geqq 0 \end{cases}$
$\dfrac{e^{-\frac{a}{p}}}{p}$	$J_0(2\sqrt{(at)})$ for Bessel functions see § 16.4
$\dfrac{e^{-\frac{a}{p}}}{\sqrt{p}}$	$\dfrac{1}{\sqrt{(\pi t)}}\cos(2\sqrt{(at)})$
$\dfrac{e^{-a\sqrt{p}}}{p}$	$\operatorname{erfc}\dfrac{a}{2\sqrt{t}} = 1 - \operatorname{erf}\dfrac{a}{2\sqrt{t}} = = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_{a/(2\sqrt{t})}^{\infty} e^{-u^2}\,du = = 1 - \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^{a/(2\sqrt{t})} e^{-u^2}\,du, \quad a \geqq 0$
$\dfrac{\cos\frac{1}{p}}{\sqrt{p}}$	$\dfrac{\cos\sqrt{(2t)}\cosh\sqrt{(2t)}}{\sqrt{(\pi t)}}$
$\dfrac{\sin\frac{1}{p}}{\sqrt{p}}$	$\dfrac{\sin\sqrt{(2t)}\sinh\sqrt{(2t)}}{\sqrt{(\pi t)}}$
$\ln\dfrac{p+a}{p}$	$\dfrac{1-e^{-at}}{t}$

If $f(t)$ is an original for the Mellin transform with the corresponding image denoted by $F(p)$ and if further

$$\int_0^{\infty} \left|f(t)t^{p_0-1}\right|\ dt < \infty$$

for some $p_0 = x_0 + iy_0$, then $f(e^{-u})\,e^{-x_0 u}$ is the original for the Fourier transform with the Fourier image $F(x_0 + iy)$. (The substitution $t = e^{-u}$ has been used.)

We will add some elementary properties of the *Fourier transform* (integrability of the functions in question is always assumed):