LAPLACE TRANSFORM

PART 1

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Laplace transform

Definition 1

Let $f(t)$ be a complex function of a real variable and let the integral $\int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t$ exists and hat finite value at least for one complex number p.

Then **Laplace transform of function *f* (*t*)** is defined by an improper integral

$$F(p) = \int_0^\infty f(t)\, e^{-pt}\,\mathrm{d}t \qquad (1)$$

Function $f(t)$ is called a **subject**.

Laplace transform is a function of complex variable p, $p = \mathrm{Re}\, p + i\, \mathrm{Im}\, p$.

Domain of Laplace transform is a set of all complex numbers for which the integral (1) is convergent.

Example 1

Laplace transform of function $f(t) = 1$
According to the definition we have:

$$\mathscr{L}\{1\} = \int_0^{\infty} 1\,\mathrm{e}^{-pt}\,\mathrm{d}t = \lim_{B\to+\infty}\int_0^{B} \mathrm{e}^{-pt}\,\mathrm{d}t = \frac{1}{-p}\lim_{B\to+\infty}\mathrm{e}^{-pB} - \frac{1}{-p}$$

If p is a complex number for which $\operatorname{Re} p > 0$, then $\lim\limits_{B\to+\infty}\mathrm{e}^{-pB} = 0$

And

$$\mathscr{L}\{1\} = \int_0^{\infty} 1\,\mathrm{e}^{-pt}\,\mathrm{d}t = \frac{-1}{p}\underbrace{\lim_{B\to+\infty}\mathrm{e}^{-pB}}_{0} - \frac{1}{-p} = \frac{1}{p}$$

Laplace transform of function $f(t) = 1$ is $\mathscr{L}\{1\} = \dfrac{1}{p}$.

Example 2

Laplace transform of function $f(t) = e^{at}$, where a is a complex number. Podle definice je

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{at}\, e^{-pt}\, dt = \lim_{B\to+\infty} \int_0^{B} e^{(a-p)t}\, dt = \frac{1}{a-p} \lim_{B\to+\infty} e^{(a-p)B} - \frac{1}{a-p}$$

If p is a complex number for which $\operatorname{Re} p > \operatorname{Re} a$ then $\lim_{B\to+\infty} e^{(a-p)B} = 0$

and

$$\lim_{B\to+\infty} \int_0^{B} e^{(a-p)t}\, dt = \frac{-1}{p-a} \underbrace{\lim_{B\to+\infty} e^{(a-p)B}}_{0} + \frac{1}{p-a} = \frac{1}{p-a}$$

Laplace transform of function e^{at} is $\mathcal{L}\{e^{at}\} = \dfrac{1}{p-a}$.

Příklad 3

Laplace transform of function $\cos t$.

$$\mathcal{L}\{\cos t\} = \int_0^\infty \cos t\, e^{-pt}\, dt = \begin{vmatrix} u = \cos t & u' = -\sin t \\ v' = e^{-pt} & v = -\dfrac{1}{p} e^{-pt} \end{vmatrix} =$$

$$= -\frac{1}{p} \cos t\, e^{-pt} \Big|_0^\infty - \frac{1}{p} \int_0^\infty \sin t\, e^{-pt}\, dt = \begin{vmatrix} u = \sin t & u' = \cos t \\ v' = e^{-pt} & v = -\dfrac{1}{p} e^{-pt} \end{vmatrix}$$

$$= -\frac{1}{p} \cos t\, e^{-pt} \Big|_0^\infty - \frac{1}{p} \left(-\frac{1}{p} \sin t\, e^{-pt} \Big|_0^\infty - \frac{1}{p} \int_0^\infty -\cos t\, e^{-pt}\, dt \right) =$$

$$\underbrace{\int_0^\infty \cos t\, e^{-pt}\, dt}_{I} = -\frac{1}{p} \cos t\, e^{-pt} \Big|_0^\infty + \frac{1}{p^2} \sin t\, e^{-pt} \Big|_0^\infty - \frac{1}{p^2} \underbrace{\int_0^\infty \cos t\, e^{-pt}\, dt}_{I}$$

Example 3 – continuation

$$I+\frac{1}{p^2}\cdot I=\frac{p^2+1}{p^2}\cdot I=-\frac{1}{p}\cos t\,\mathrm{e}^{-pt}\Big|_0^\infty+\frac{1}{p^2}\sin t\,\mathrm{e}^{-pt}\Big|_0^\infty$$

If p is a complex number for which $\operatorname{Re} p>0$ then

$$\lim_{B\to+\infty}\cos t\,\mathrm{e}^{-pB}=0,\ \lim_{B\to+\infty}\sin t\,\mathrm{e}^{-pB}=0$$

and

$$\frac{p^2+1}{p^2}\cdot I=-\frac{1}{p}(0-1)+\frac{1}{p^2}(0-0)\Rightarrow I=\frac{p}{p^2+1}$$

Laplace transform of function $\cos t$ is $\mathscr{L}\{\cos t\}=\dfrac{p}{p^2+1}$.

Example 4

According the definition we can find Laplace transform of $\sin t$

$$\mathcal{L}\{\sin t\} = \int_0^\infty \sin t \, e^{-pt} \, dt = \ldots \text{ 2 times by parts } \ldots$$

$$= -\frac{1}{p} \sin t \, e^{-pt} \Big|_0^\infty + \frac{1}{p^2} \cos t \, e^{-pt} \Big|_0^\infty - \frac{1}{p^2} \int_0^\infty \sin t \, e^{-pt} \, dt$$

$$\underbrace{\int_0^\infty \sin t \, e^{-pt} \, dt}_{I} = -\frac{1}{p} \sin t \, e^{-pt} \Big|_0^\infty + \frac{1}{p^2} \cos t \, e^{-pt} \Big|_0^\infty - \frac{1}{p^2} \underbrace{\int_0^\infty \sin t \, e^{-pt} \, dt}_{I}$$

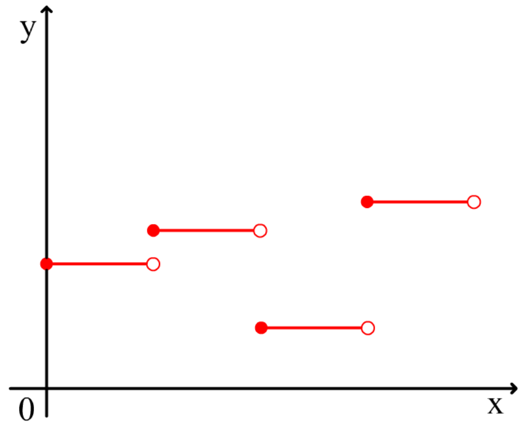
Analogosly, if $\operatorname{Re} p > 0$, we obtain

$$\frac{p^2+1}{p^2} \cdot I = -\frac{1}{p}(0-0) + \frac{1}{p^2}(1-0) \Rightarrow I = \frac{1}{p^2+1}$$

Laplace transform of function $\sin t$ is $\mathcal{L}\{\sin t\} = \dfrac{1}{p^2+1}$.

Piecewise continuous function

Definition 2 Function f has at a point x **discontinuity of 1st kind** if it is not continuous at x, but it has one sided finite limits at x (they may differ).



Definition 3
Function $f(t)$ defined on $\langle a,b\rangle,\ a,b \in \mathrm{R}$, is called piecewise continuous, if f has at $\langle a,b\rangle$ only finite number of points with discontinuity of 1st kind.

Accumulated point

Definition 4

A point x is called **an accumulated point of set *M*,** if in each neighborhood there exists at least one point of set M different from x .

Examples

1. Set of natural numbers has only one accumulated point plus infinity.
2. Set $\left\{ -1^{\,n} + \frac{1}{n} \right\}, n = 1, 2, 3, ...$ has two accumulated points -1 a 1.
3. Open interval (a,b) has infinitely many accumulated points in R: every point within interval (a,b) and also a and b.

Function of exponential order

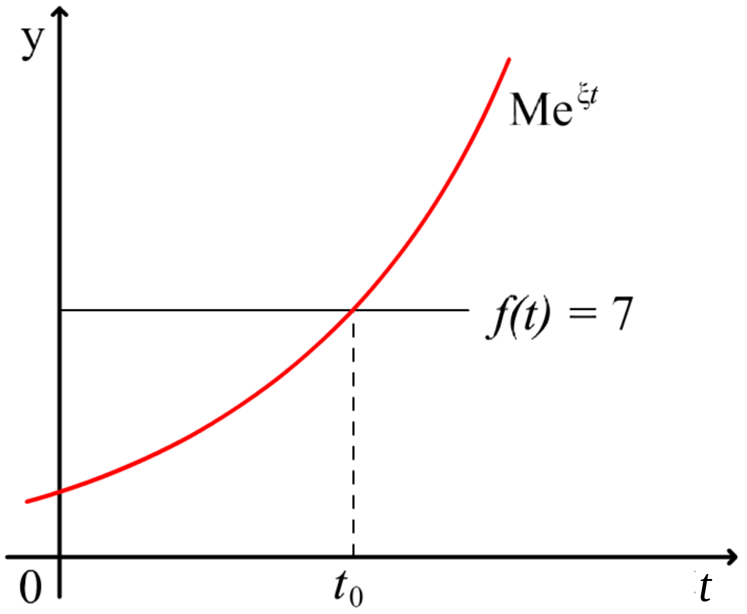
Definition 5 Function $f(t)$ is of exponential order with growth index $\boldsymbol{\xi_1}$, if point $+\infty$ is accumulated point of its domain and if there exist real numbers čí t_0 and $M > 0$ so that the inequality

$$\left| f(t) \right| \leq M\, \mathrm{e}^{\xi_1 t}$$

holds for every $t > t_0$, for which $f(t)$ is defined.

We write $f(t) = \mathcal{O}\left(\mathrm{e}^{\xi_1 t}\right)$ (symbol of a function of exponential order)

Example



Function of Exponential Order

Remark:

If number ξ_1 is growth index of function f, then also $\xi_1' > \xi_1$ is its growth index.

Every function of exponential order has infinitely many growth indices.

Usually, we use the smallest one.

Examples of functions of exponential order:

a) Every bounded function $f(t)$ (defined for big enough values of t) is of exponential order with growth index 0.

b) Function e^{at}, a real is of exponential order with growth index a.

c) Function t^n, n non-negative, is of exponential order with growth index equal to any positive number.

Subject of Standard type

Definition 6

Complex function $f(t)$ of real variable t is called **subject of standard type**, if it has the following three properties:

1. $f(t)$ is piecewise continuous on $\langle 0, +\infty)$.
2. $f(t)$ is of exponential order.
3. $f(t)$ is equal to zero for all $t < 0$.

Theorem On the Existence of LT

Theorem 1

Let $f(t)$ be a subject of standard type(SST) and ξ_1 its growth index. Then Laplace transform

$$F(p) = \mathfrak{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-pt}\, dt$$

exists and the integral $\int_0^\infty f(t)\, e^{-pt} dt$ is absolutely convergent for all complex p, for which $Re\, p > \xi_1$.

(So, Laplace transform is defined in a half-plane $Re\, p > \xi_1$).

Remark: The above theorem states sufficient condition of existence and unicity of LT. There might exist functions, that have Laplace transform, but are not SST.

ON LINEARITY OF LT

Theorem 2

Let $f_i(t)$ are SSTs and let $\mathrm{F}_i(p)$, $i = 1, 2, 3, \ldots, n$ are respective LTs.
Let a_i are auxiliary complex constants.
Then pro for all complex numbers p, for which $\mathrm{F}_i(p)$ are defined,
The following identity holds:

$$\mathscr{L}\left\{\sum_{i=1}^{n} a_i f_i(t)\right\} = \sum_{i=1}^{n} a_i F_i(p)$$

Important!

The theorem holds only upon the assumption of finite number n of subjects of standard type.

Linearity of LT - Examples

$$\mathcal{L}\left\{3\cos t - 5e^{-2t}\right\} = 3\mathcal{L}\left\{\cos t\right\} - 5\mathcal{L}\left\{e^{-2t}\right\} =$$

$$= 3\frac{p}{p^2+1} - 5\frac{1}{p+2}$$

$$\mathcal{L}\left\{-4 + 7t - 3\sin 2t\right\} = -4\mathcal{L}\left\{1\right\} + 7\mathcal{L}\left\{t\right\} + 3\mathcal{L}\left\{\sin 2t\right\} =$$

$$= -4\frac{1}{p} + 7\frac{1}{p^2} - 3\frac{2}{p^2+4}$$

Change of scale

Theorem 3 Let $f(t)$ is SST $\mathfrak{L}\{f(t)\}=F(p)$ and let k is positive constant. Then function $g(t)=f(kt)$ is also SST and the following identities hold:

$$\mathfrak{L}\{g(t)\}=\mathfrak{L}\{f(kt)\}=\frac{1}{k}F\left(\frac{p}{k}\right)$$

$$\mathfrak{L}\left\{f\left(\frac{t}{k}\right)\right\}=kF(kp)$$

Shift in LT

Theorem 4 Let $f(t)$ is SST is of exponential order $\alpha, \mathcal{L}\{f(t)\} = F(p)$ and let $a \in R$.

Then:

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(p-a) \qquad p > a + \alpha$$

Proof: Both functions $f(t)$ and e^{at} are SSTs so is their product. Using the definition we get:

$$\mathcal{L}\left\{e^{at} \cdot f(t)\right\} = \int_0^\infty f(t) e^{at} e^{-pt} \, dt = \int_0^\infty f(t) e^{-(p-a)t} \, dt = F(p-a)$$

Shift in LT- Examples

Laplace transformt of function e$^{5t} \cdot \sin(2t)$

$$\mathfrak{L}\{\sin(\omega t)\} = \frac{\omega}{p^2+\omega^2} \quad \Rightarrow \quad \mathfrak{L}\{\sin(2t)\} = \frac{2}{p^2+4}$$

$$\mathfrak{L}\{e^{at}\sin(\omega t)\} = \frac{\omega}{(p-a)^2+\omega^2} \quad \Rightarrow \quad \mathfrak{L}\{e^{5t}\sin(2t)\} = \frac{2}{(p-5)^2+4}$$

On Derivative of LT

Theorem 5 Let $f(t)$ be SST of exponential order a, $\mathcal{L}\{f(t)\} = F(p)$. .
Then:

$$\mathcal{L}\{t \cdot f(t)\} = -F'(p) = -\frac{\mathrm{d}}{\mathrm{d}\,p}\big(F(p)\big) \qquad p > a$$

Remark: As function $t \cdot f(t)$ is also SST,
the above theorem can be used for function $t \cdot f(t)$ as well.

$$\mathcal{L}\{t \cdot t \cdot f(t)\} = -\frac{d}{dp}\mathcal{L}\{t \cdot f(t)\} = \frac{\mathrm{d}^2}{\mathrm{d}\,p^2}\big(F(p)\big)$$

By means of mathematical induction the following formula can be proven:

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\,p^n}\big(F(p)\big), \quad n \text{ natural number.}$$

Derivative of LT - Examples

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{p-a} \qquad \mathcal{L}\left\{t\,\mathrm{e}^{at}\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,p}\left(\frac{1}{p-a}\right) = \frac{1}{(p-a)^2}$$

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{p^2+\omega^2} \qquad \mathcal{L}\left\{t\,\sin(\omega t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,p}\left(\frac{\omega}{p^2+\omega^2}\right) = \frac{2p\omega}{(p^2+\omega^2)^2}$$

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \frac{p}{p^2+\omega^2} \qquad \mathcal{L}\left\{t\,\cos(\omega t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,p}\left(\frac{p}{p^2+\omega^2}\right) = \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$$

Inverse Laplace transform

Inverse LT is not uniquely defined. According to definition, two functions, that differ only in finite number of points have identical integral.
Under the assumption that functions that are SST with values at points of discontinuity defined by

$$f(t) = \lim_{\varepsilon \to 0+} \frac{f(t+\varepsilon) + f(t-\varepsilon)}{2}.$$

Lerch theorem (On uniqueness) holds.
Simply speaking:
To the given $F(p)$ there exists at least one subject of standard type $f(t)$ with the property defined above that has LT $\mathscr{L}(f(t)) = F(p)$.
Such a subject is denoted by : $\mathscr{L}^{-1}F(p)$.

Remark: Analogous statements hold for inverse LT.

INVERSE LT OF RATIONAL FUNCTION

Theorem 6
Necessary and sufficient condition for a rational function $F(p)$ to be A Laplace transform of a SST is that the function F(p) is strictly rational function.
In other words: $F(p)$ must have the order of numerator strictly smaller than the order of denominator.

How the subject of the strictly rational function can be found?

By factoring the function into a sum of partial fractions and then using inverse LT the origin (subject) can be found.

Inverse LT

Find SST to the Laplace transform $F(p)=\dfrac{p+2}{p^3-p}$

$$F(p)=\frac{p+2}{p^3-p}=\frac{p+2}{p(p+1)(p-1)}=\frac{A}{p}+\frac{B}{p+1}+\frac{C}{p-1}$$

$$\Rightarrow A=-2,\quad B=\frac{1}{2},\quad C=\frac{3}{2}$$

$$F(p)=-2\frac{1}{p}+\frac{1}{2}\frac{1}{p+1}+\frac{3}{2}\frac{1}{p-1}$$

$$f(t)=\mathfrak{L}^{-1}\{F(p)\}=-2\mathfrak{L}^{-1}\left\{\frac{1}{p}\right\}+\frac{1}{2}\mathfrak{L}^{-1}\left\{\frac{1}{p+1}\right\}+\frac{3}{2}\mathfrak{L}^{-1}\left\{\frac{1}{p-1}\right\}=$$

$$=-2\cdot 1\quad+\quad\frac{1}{2}e^{-t}\quad+\quad\frac{3}{2}e^{t}$$

ON t–DERIVATIVE RULE

Theorem 7 Let $f'(t)$ is SST of exponential order α, $\mathfrak{L}\{f(t)\} = F(p)$.

Then .

$$\mathfrak{L}\{f'(t)\} = pF(p) - f(0_+), \quad p > \max(0, \alpha)$$

Proof:

$$\mathfrak{L}\{f'(t)\} = \int_0^\infty f'(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t = \begin{vmatrix} u = \mathrm{e}^{-pt} & u' = -p\,\mathrm{e}^{-pt} \\ v' = f'(t) & v = f(t) \end{vmatrix} =$$

$$\left[\mathrm{e}^{-pt} f(t)\right]_0^{+\infty} - \int_0^\infty p\, f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t = -f(0_+) + p\mathfrak{L}\{f(t)\}$$

On t–Derivative Rule Consequence

Let $f^{(n)}(t)$ is SST of exponential order $\alpha, \mathfrak{L}\left\{f(t)\right\}=F(p)$.

Then for $p>\max(\alpha,0)$ the following identity holds .

$$\mathfrak{L}\left\{f^{(n)}(t)\right\}=p^{n}F(p)-p^{n-1}f(0_{+})-p^{n-2}f'(0_{+})-\ldots-p\,f^{(n-2)}(0_{+})-f^{(n-1)}(0_{+})$$

Proof: Using of mathematical induction.

On t–Derivative Rule – An Example

$$y'' + 5y' = 5, \quad p.p. \quad y(0+) = 1, \quad y'(0+) = 0$$

$$p^2 Y(p) - p \cdot y(0+) - y'(0+) + 5\left(pY(p) - y(0+)\right) = \frac{5}{p}$$

$$p^2 Y(p) - p + 5\left(pY(p) - 1\right) = \frac{5}{p}$$

$$p(p+5)Y(p) = \frac{5}{p} + p + 5$$

$$Y(p) = \frac{p^2 + 5p + 5}{p^2(p+5)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+5} \Rightarrow \quad A = \frac{4}{5}, B = 1, C = \frac{1}{5}$$

$$Y(p) = \frac{4}{5}\frac{1}{p} + \frac{1}{p^2} + \frac{1}{5}\frac{1}{p+5} \quad \Rightarrow \quad y(t) = \frac{4}{5} + t + \frac{1}{5}e^{-5t}$$

Initial Value Problem of ODE with constant coefficients – An Example

$$y'' + y = \cos t, \quad p.p. \quad y(0+) = -1, \quad y'(0+) = 1$$

$$p^2 Y(p) - p \cdot y(0+) - y'(0+) + Y(p) = \frac{p}{p^2+1}$$

$$\left(p^2+1\right) Y(p) = \frac{p}{p^2+1} - p + 1$$

$$Y(p) = \frac{p}{\left(p^2+1\right)^2} - \frac{p-1}{p^2+1} = \frac{p}{\left(p^2+1\right)^2} - \frac{p}{p^2+1} + \frac{1}{p^2+1}$$

$$Y(p) = \frac{1}{2}\frac{2p}{\left(p^2+1\right)^2} - \frac{p}{p^2+1} + \frac{1}{p^2+1} \quad \Rightarrow \quad y(t) = \frac{1}{2} t \sin t - \cos t + \sin t$$

On t–integral Rule

Theorem 8 Let $f(t)$ is SST of exponential order α, $\mathfrak{L}\{f(t)\} = F(p)$.

Then
$$\mathfrak{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = \frac{F(p)}{p}, \quad p > \max(0,\alpha)$$

Proof:

$$g(t) = \int_0^t f(u)\,\mathrm{d}u \quad \Rightarrow \quad \mathfrak{L}\{g(t)\} = \mathfrak{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = G(p)$$

$$g'(t) = f(t), \qquad g(0_+) = \int_0^0 f(u)\,\mathrm{d}u = 0$$

$$F(p) = \mathfrak{L}\{f(t)\} = \mathfrak{L}\{g'(t)\} = pG(p) - g(0_+) = pG(p)$$

$$G(p) = \frac{F(p)}{p}$$

On t–integral Rule – An Example

Solve integral equation:

$$x(t) - 2\int_0^t x(u)\,\mathrm{d}u = \sin t$$

$$X(p) - 2\frac{X(p)}{p} = \frac{1}{p^2+1}$$

$$X(p)\left(1 - \frac{2}{p}\right) = \frac{1}{p^2+1}$$

$$X(p)\left(\frac{p-2}{p}\right) = \frac{1}{p^2+1} \Rightarrow X(p) = \frac{p}{\left(p^2+1\right)\left(p-2\right)}$$

$$X(p) = \frac{A}{p-2} + \frac{Bp+C}{p^2+1} \Rightarrow X(p) = \frac{\frac{2}{5}}{p-2} + \frac{-\frac{2}{5}p}{p^2+1} + \frac{\frac{1}{5}}{p^2+1}$$

$$x(t) = \tfrac{2}{5}\mathrm{e}^{2t} - \tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t$$

The integral equation can be differentiated – the result is differential equation with initial condition.

$$x(t)-2\int_0^t x(u)\,\mathrm{d}u=\sin t \rightarrow x'(t)-2x(t)=\cos t,\quad x(0)=0$$

$$pX(p)-x(0_+)-2X(p)=\frac{p}{p^2+1}$$

$$X(p)(p-2)=\frac{p}{p^2+1}$$

$$X(p)=\frac{p}{(p^2+1)(p-2)}$$

$$X(p)=\frac{A}{p-2}+\frac{Bp+C}{p^2+1}$$

$$X(p)=\frac{\frac{2}{5}}{p-2}+\frac{-\frac{2}{5}p}{p^2+1}+\frac{\frac{1}{5}}{p^2+1}$$

$$x(t)=\tfrac{2}{5}\mathrm{e}^{2t}-\tfrac{2}{5}\cos t+\tfrac{1}{5}\sin t$$

On t–integral Rule – An Example

Solve integro-diferential equation

$$x' - 4x + 3\int_0^t x(u)\,\mathrm{d}u = 2\,\mathrm{e}^t, \quad \text{with initial condition } x(0_+) = -1$$

$$pX(p) + 1 - 4X(p) + 3\frac{X(p)}{p} = 2\frac{1}{p-1}$$

$$X(p)\left(p - 4 + \frac{3}{p}\right) = \frac{2}{p-1} - 1$$

$$X(p)\left(\frac{p^2 - 4p + 3}{p}\right) = \frac{-p+3}{p-1} \Rightarrow X(p) = \frac{-p}{(p-1)^2}$$

$$X(p) = \frac{-1}{p-1} + \frac{-1}{(p-1)^2}$$

$$x(t) = -1\mathrm{e}^t - t\,\mathrm{e}^t$$

SOLVING CAUCHY PROBLEM FOR LINEAR ODE WITH CONSTANT COEFFICIENS - EXAMPLE

$$y''(x)+5y'(x)+6y(x)=4e^{-x}, \quad p.p. \quad y(0+)=0, \quad y'(0+)=0$$

$$p^2Y(p)-p\cdot y(0+)-y'(0+)+5\left(pY(p)-y(0+)\right)+6Y(p)=\frac{4}{p+1}$$

$$\left(p^2+5p+6\right)Y(p)=\frac{4}{p+1}$$

$$(p+2)(p+3)Y(p)=\frac{4}{p+1}$$

$$Y(p)=\frac{4}{(p+1)(p+2)(p+3)}$$

$$Y(p)=2\frac{1}{p+1}-4\frac{1}{p+2}+2\frac{1}{p+3} \Rightarrow y(t)=2\mathrm{e}^{-x}-4\mathrm{e}^{-2x}+2\mathrm{e}^{-3x}$$